Potential scattering with field discontinuities at the boundaries

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We discuss the validity of the approximate scalar theories for scattering of electromagnetic waves, which implicitely assume that the field and its normal derivative are continuous at the the scatterer's boundary. However, as is well known certain components of the electromagnetic field are discontinuous across such a boundary. We present a modified formalism that takes these discontinuities into account. We show that in some case the total scattering cross section may differ appreciably from that predicted by the usual theory.
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I. INTRODUCTION

Because of the complexity of the theory of scattering of electromagnetic waves on macroscopic bodies, the situation is often highly idealized. A frequently used idealization considers the scattering body to be a perfectly conducting medium which, consequently, gives rise to a strong reflected wave. While this approximation is often adequate with microwaves and radio waves, it is seldom appropriate in the optical region of the electromagnetic spectrum. For example, when one considers the scattering of a light wave at an aperture, the material of the screen containing the aperture is frequently better approximated by assuming that the screen is highly absorbing rather than that it is highly reflecting. The difference between these two situations is far reaching, because in the case of a perfectly conducting medium one is led to a boundary value problem, whereas in all other cases one encounters a saltus problem, i.e., a problem involving jumps of the electromagnetic field vectors at the boundary of the medium.

Another approximation often used consists of ignoring the polarization properties of the field by treating it as a scalar field. While the scalar problem can often provide some insight into the nature of the scattered field, its use disguises the following difficulty: As is well known some of the components of the electromagnetic field vectors are discontinuous across boundaries of the scattering bodies [1]. The field discontinuities themselves depend on the generally unknown field. It is this feature which makes electromagnetic scattering problems much more complicated than the quantum-mechanical problems of potential scattering. To our knowledge the implications of the discontinuities in the field variable at sharp boundaries of scattering bodies have not been previously investigated in the scalar model, except in Kottler's theory of diffraction at a black screen [2]. In the present paper we investigate how the usual scalar integral equation formalism can be extended to take into account possible field discontinuities. It is to be noted that in Kottler's theory the field discontinuities are prescribed whereas in our approach they are not. While we do not address the question of how to determine the actual discontinuities, we obtain some quantitative results which indicate that in some cases the effect may be significant even at large distances from the scatterer. The relationship between boundary value problems and saltus problems is discussed in Ref. [3].

In Sec. II of this paper some relations are noted which involve the discontinuities of electromagnetic field vectors for scattering by bodies with sharp boundaries. In Sec. III we consider scalar scattering of a plane wave by a homogeneous sphere when the field and/or its normal derivative suffer a discontinuity on its boundary surface. We show how the discontinuities affect the scattering cross section and we illustrate the results by a numerical example. In Sec. IV we obtain a generalized integral equation of potential scattering which includes the effects of discontinuities of the field and of its normal derivative on the boundary.

II. BEHAVIOR OF THE ELECTROMAGNETIC FIELD ACROSS THE BOUNDARY OF THE SCATTERER

Consider the two-dimensional scattering configuration which is depicted in Fig. 1. A current sheet distribution $\mathbf{J} = (J_x, 0, 0)$ which is situated in free space generates an elec-

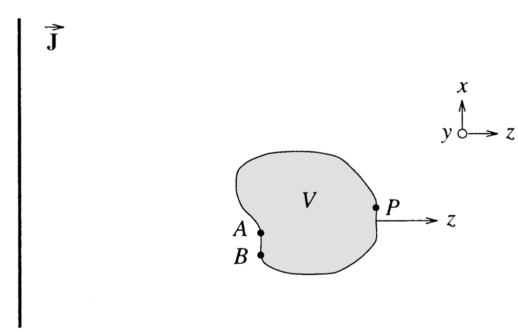

FIG. 1. A two-dimensional scattering configuration. A current sheet (infinite in the x and y directions) situated in free space, with $\mathbf{J} = (J_x, 0, 0)$, generates an electromagnetic field that is scattered by the volume V which has permittivity ϵ. The points A, B, and P lie on the boundary of the scattering volume.

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tromagnetic field $\mathbf{E}=(E_x,0,0)$ and $\mathbf{H}=(0,H_y,0)$. The field is scattered by an isotropic, homogeneous and nonconducting medium V with permittivity ϵ. Both the current distribution and the scatterer are assumed to be infinite in the y direction. This implies that the field is independent of y. In steady state, with time dependence $\exp(-i\omega t)$, the x component of the first Maxwell equation in SI units reduces to

$$\partial_z H_y = i\omega\epsilon(x,z)E_x, \tag{2.1}$$

in source-free regions. The point P at the boundary of the scatterer is chosen so that the outward normal coincides with the direction of z. P lies in the plane $z=0$. Since both H_y and E_x are tangential to the boundary at P, they are continuous there. However, it follows immediately from Eq. (2.1) that the derivative $\partial_z H_y$ is discontinuous at P. If we define

$$\left(\frac{\partial H_y}{\partial z}\right)^- \equiv \lim_{z\to -0}\frac{\partial H_y}{\partial z} = i\omega\epsilon E_x(P), \tag{2.2}$$

$$\left(\frac{\partial H_y}{\partial z}\right)^+ \equiv \lim_{z\to +0}\frac{\partial H_y}{\partial z} = i\omega\epsilon_0 E_x(P), \tag{2.3}$$

then

$$\Delta\left(\frac{\partial H_y}{\partial z}\right) \equiv \left(\frac{\partial H_y}{\partial z}\right)^+ - \left(\frac{\partial H_y}{\partial z}\right)^- \tag{2.4}$$

$$= i\omega E_x(P)(\epsilon_0-\epsilon) = \left(\frac{\epsilon_0}{\epsilon}-1\right)\left(\frac{\partial H_y}{\partial z}\right)^-. \tag{2.5}$$

We conclude that in this configuration the field component H_y is continuous at P, whereas the normal derivative $\partial H_y(P)/\partial z$ has a discontinuity.

Next consider the boundary points $A=(x_1,0,z)$ and $B=(x_2,0,z)$ in Fig. 1. The portion of the boundary between them is parallel to the x axis. Since E_x is tangential to the boundary,

$$E_x(A)^+ = E_x(A)^-, \tag{2.6}$$

$$E_x(B)^+ = E_x(B)^-, \tag{2.7}$$

where, as before, + and − indicate that the limit is taken from outside and from inside the boundary of the scatterer, respectively. One has

$$E_x(B)^+ - E_x(A)^+ = (x_2-x_1)\left(\frac{\partial E_x(B)}{\partial x}\right)^+ + \cdots, \tag{2.8}$$

$$E_x(B)^- - E_x(A)^- = (x_2-x_1)\left(\frac{\partial E_x(B)}{\partial x}\right)^- + \cdots. \tag{2.9}$$

Since Eqs. (2.8) and (2.9) hold for any distance x_2-x_1, it follows that

$$\left(\frac{\partial E_x}{\partial x}\right)^+ = \left(\frac{\partial E_x}{\partial x}\right)^-, \tag{2.10}$$

for any boundary point between A and B. In addition, we have at these points

$$(\nabla\cdot\mathbf{D})^+ = (\nabla\cdot\mathbf{D})^- = 0. \tag{2.11}$$

where $\mathbf{D}(x,z)=\epsilon(x,z)\mathbf{E}(x,z)$ is the electric displacement vector. Hence,

$$\epsilon_0\left[\left(\frac{\partial E_x}{\partial x}\right)^+ + \left(\frac{\partial E_z}{\partial z}\right)^+\right] = 0, \tag{2.12}$$

$$\epsilon\left[\left(\frac{\partial E_x}{\partial x}\right)^- + \left(\frac{\partial E_z}{\partial z}\right)^-\right] = 0. \tag{2.13}$$

Since $\epsilon_0\neq 0$ and $\epsilon\neq 0$, we have, if we use Eq. (2.10),

$$\left(\frac{\partial E_z}{\partial z}\right)^+ = \left(\frac{\partial E_z}{\partial z}\right)^-. \tag{2.14}$$

By using the well-known ''pill box'' construction (Ref. [1], Sec. 1.1.3), it can be shown that for the normal component of the electric field at points between A and B

$$\epsilon_0 E_z^+ = \epsilon E_z^-, \tag{2.15}$$

or, equivalently,

$$E_z^+ - E_z^- = \left(\frac{\epsilon}{\epsilon_0}-1\right)E_z^-. \tag{2.16}$$

We conclude that in this configuration the outside limit of the normal derivative $\partial E_z/\partial z$ at points that lie between A and B is equal to the inside limit, whereas the field component E_z has a discontinuity there.

III. PLANE WAVE SCATTERING BY A HOMOGENEOUS SPHERE IN THE PRESENCE OF DISCONTINUITIES

As an example of how a field discontinuity can affect the far field and the cross section in a scattering process, we consider the scattering of a scalar plane wave by a homogeneous sphere. We use the method of partial waves which is described, for example, in Refs. [4–6]. Our objective is to obtain an expression for the phase shift, from which the scattering amplitude and the total scattering cross section can be determined.

We assume that both the field U and its derivative $\partial U/\partial n$ along the outward normal $\vec{n}$ have a finite discontinuity at the boundary of the scatterer:

$$\Delta U \equiv U^+(\mathbf{r}) - U^-(\mathbf{r}), \tag{3.1}$$

$$\Delta\left(\frac{\partial U}{\partial n}\right) \equiv \left(\frac{\partial U(\mathbf{r})}{\partial n}\right)^+ - \left(\frac{\partial U(\mathbf{r})}{\partial n}\right)^-. \tag{3.2}$$

As before, the superscripts + and − indicate limiting values taken from outside and inside of the scatterer, respectively. Also, within the accuracy of the linear theory, both ΔU and $\Delta(\partial U/\partial n)$ are proportional to U^- and $(\partial U/\partial n)^-$, respectively, i.e.,

$$\Delta U = \alpha U^-(\mathbf{r}), \tag{3.3}$$

$$\Delta\left(\frac{\partial U}{\partial n}\right) = \beta\left(\frac{\partial U(\mathbf{r})}{\partial n}\right)^-, \tag{3.4}$$

where α and β are constants. [In the first example of Sec. II $\alpha=0$, $\beta=(\epsilon_0/\epsilon-1)$, and in the second example $\alpha=(\epsilon/\epsilon_0-1)$, $\beta=0$.]

Let us consider a homogeneous scattering potential F of the form

$$F(r)=\begin{cases} F_0 & \text{if } r<R, \\ 0 & \text{if } r>R, \end{cases} \tag{3.5}$$

where F_0 is a real-valued constant. We expand the field in spherical harmonics. We assume that the incident plane wave propagates along the z axis. The azimuthal dependence then drops out, and we have

$$U_k(r,\theta)=\sum_{l=0}^{\infty} a_l(k)U_{l,k}(r)P_l(\cos\theta), \tag{3.6}$$

with $a_l(k)$ being expansion coefficients, k the free space wave number, $U_{l,k}(r)$ radial functions, and $P_l(\cos\theta)$ Legendre polynomials. Inside the scatterer, $U_{l,k}$ is taken to be proportional to the spherical Bessel function $j_l(k_0r)$:

$$U_{l,k}(r)=C_l j_l(k_0r) \quad (r<R), \tag{3.7}$$

with

$$k_0^2\equiv k^2+F_0 \tag{3.8}$$

and C_l are constants. Outside the scatterer, the field may be expressed as a superposition of spherical Bessel functions $j_l(kr)$ and spherical Neumann functions $n_l(kr)$:

$$U_{l,k}(r)=j_l(kr)-\tan\delta_l n_l(kr) \quad (r>R). \tag{3.9}$$

The constant C_l and the phase shift δ_l are determined by matching the interior solution (3.7) and the exterior solution (3.9) at $r=R$, taking the boundary discontinuities (3.1) and (3.2) into account. From the phase shift, the scattering amplitude $f(\theta,k)$, the differential scattering cross section $d\sigma(\theta,k)/d\Omega$ and the total scattering cross section $\sigma_{\text{tot}}(k)$ can be calculated, and one finds that

$$f(\theta,k)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)(e^{2i\delta_l}-1), \tag{3.10}$$

$$\frac{d\sigma(\theta,k)}{d\Omega}=\frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}\times(2l+1)(2l'+1)P_l(\cos\theta)P_{l'}(\cos\theta), \tag{3.11}$$

$$\sigma_{\text{tot}}(k)=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{3.12}$$

Our Eqs. (3.10)–(3.12) are Eqs. (4.63), (4.68), and (4.71) of Ref. [6].

On matching the interior solution Eq. (3.7) and the exterior solution Eq. (3.9) at $r=R$ while taking the discontinuity parameters α and β into account [see Eqs. (3.3) and (3.4)], we find that

$$(1+\alpha)C_l j_l(k_0R)=j_l(kR)-\tan\delta_l n_l(kR), \tag{3.13}$$

and

$$(1+\beta)C_l k_0 j_l'(k_0R)=k[j_l'(kR)-\tan\delta_l n_l'(kR)]. \tag{3.14}$$

Here

$$j_l'(k_0R)=\left.\frac{dj_l(x)}{dx}\right|_{x=k_0R}, \tag{3.15}$$

$$j_l'(kR)=\left.\frac{dj_l(x)}{dx}\right|_{x=kR}, \tag{3.16}$$

$$n_l'(kR)=\left.\frac{dn_l(x)}{dx}\right|_{x=kR}. \tag{3.17}$$

Dividing Eq. (3.13) by Eq. (3.14) and defining the parameters

$$\gamma_l\equiv k_0\frac{j_l'(k_0R)}{j_l(k_0R)}, \tag{3.18}$$

$$\rho(\alpha,\beta)\equiv\frac{1+\beta}{1+\alpha}, \tag{3.19}$$

one readily finds that

$$\tan\delta_l=\frac{kj_l'(kR)-\rho(\alpha,\beta)\gamma_l j_l(kR)}{kn_l'(kR)-\rho(\alpha,\beta)\gamma_l n_l(kR)}. \tag{3.20}$$

It follows on comparison of these results with those of Ref. [6] for the case when $\alpha=\beta$, that the field discontinuities will have no influence on the phase shift, and hence on the scattered field. We recall that for $l=0$

$$j_0(x)=\frac{\sin x}{x}, \quad n_0(x)=-\frac{\cos\chi}{x}. \tag{3.21}$$

If we also choose $\alpha=\beta$, then Eq. (3.20) reduces to

$$\tan\delta_0=\frac{k\tan(k_0R)-k_0\tan(kR)}{k\tan(kR)\tan(k_0R)+k_0}, \tag{3.22}$$

which is Eq. (4.154) of Ref. [6].

The relative size of the scatterer determines the number of partial waves over which the summation in Eq. (3.6) has to be carried out. Significant scattering will only occur for those waves for which $l\leqslant kR$ (see, e.g., Ref. [6], p. 73).

Equation (3.20) is an expression for δ_l in terms of the discontinuity parameters α and β. If, because of the discontinuities that are now taken into account, the phase shift changes, the scattering amplitude will also change, and so will the total scattering cross section which depend on δ_l through Eqs. (3.10) and (3.12), respectively. A numerical example for a homogeneous scattering potential is presented in Fig. 2. It indicates how the total scattering cross section σ_{tot} depends on the the discontinuity parameters α and β. The ranges of α and β correspond to $1<\epsilon/\epsilon_0<4$ [cf. the expressions for α and β below Eq. (3.4)]. Over these ranges,

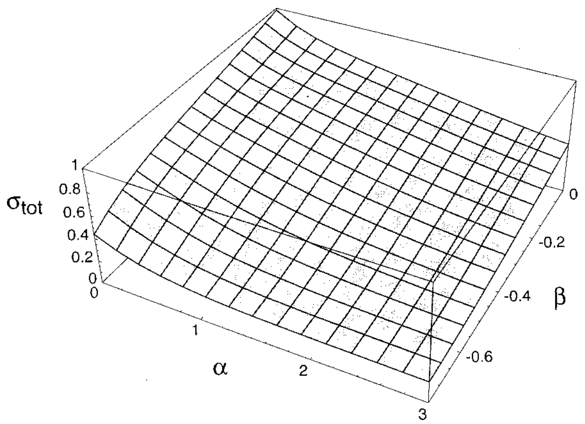

FIG. 2. The total scattering cross section $\sigma_{\rm tot}$, given by Eqs. (3.12) and (3.20), for scattering of a plane scalar wave on a spherical potential, as a function of the discontinuity parameters α and β, with $k=3\times10^7\ {\rm m}^{-1}$, $k_0=5\times10^7\ {\rm m}^{-1}$, and $R=8\times10^{-8}$ m. The total scattering cross section is expressed in units of $10^{-13}\ {\rm m}^2$.

the total scattering cross section varies by about a factor of 5. In this example the first seven partial waves were taken into account. Higher-order contributions were found to be negligible.

IV. SCALAR SCATTERING THEORY

In this section we develop a scalar potential scattering theory that takes field discontinuities at boundaries into account. Consider a scatterer of constant refractive index n, occupying a volume V, in free space (see Fig. 3). The exterior volume is denoted as $\tilde{V}$. The time-harmonic field U satisfies the Helmholtz equation

$$(\nabla^2+k^2)U(\mathbf{r})=0 \quad (\mathbf{r}\in\tilde{V}), \tag{4.1}$$

$$(\nabla^2+n^2k^2)U(\mathbf{r})=0 \quad (\mathbf{r}\in V). \tag{4.2}$$

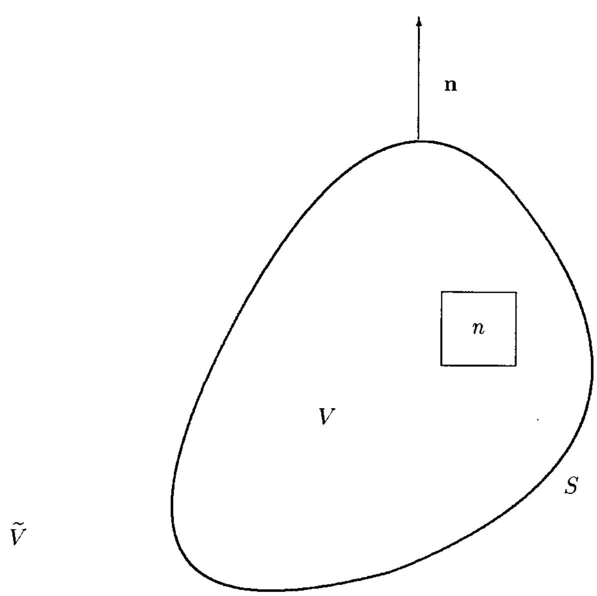

FIG. 3. Scattering volume V containing a homogeneous medium with refractive index n. The volume is bounded by the surface S and $\mathbf{n}$ is the unit outward normal. The exterior volume is denoted by $\tilde{V}$.

Here k is the free-space wave number and a time-dependent factor $\exp(-i\omega t)$ has been suppressed. We introduce a scattering potential F by rewriting Eq. (4.2) in the form

$$(\vec{\nabla}^2+k^2)U(\mathbf{r})=-4\pi F(\mathbf{r})U(\mathbf{r}), \tag{4.3}$$

with

$$F(\mathbf{r})\equiv\frac{k^2}{4\pi}[n^2(\mathbf{r})-1], \tag{4.4}$$

and

$$n(\mathbf{r})=\begin{cases} n_0 & \text{if } \mathbf{r}\in V,\\ 1 & \text{if } \mathbf{r}\in\tilde{V},\end{cases} \tag{4.5}$$

where n_0 is a real-valued constant. The Green function G of the Helmholtz operator satisfies the equation

$$(\vec{\nabla}^2+k^2)G(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{4.6}$$

We choose G to be an outgoing spherical wave, i.e.,

$$G(\mathbf{r},\mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.7}$$

Let $\hat{\mathbf{n}}'$ be the outward unit normal on a surface S' which bounds a domain of volume V'. It should be noted that we no longer restrict our analysis to a spherical scatterer, i.e., its shape is now arbitrary. By using Green's theorem and Eqs. (4.3) and (4.6) one finds that

$$-4\pi\int_{V'}\delta(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')d^3r' +4\pi\int_{V'}G(\mathbf{r},\mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')d^3r'=\Sigma'(\mathbf{r}), \tag{4.8}$$

with

$$\Sigma'(\mathbf{r})\equiv\int_{S'}\left(U(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}-G(\mathbf{r},\mathbf{r}')\frac{\partial U(\mathbf{r}')}{\partial n'}\right)dS'. \tag{4.9}$$

The volume V' is taken either as $V^<$ or $V^>$ which denote the domains V and $\tilde{V}$, respectively, without their boundary surface S. Following the analysis of Ref. [7] we choose the observation point $\mathbf{r}$ to lie either within $V^<$ (then $\mathbf{r}=\mathbf{r}_<$), or within $V^>$ (then $\mathbf{r}=\mathbf{r}_>$). The four possible combinations and the resulting forms of Eq. (4.8) are

(1) $\mathbf{r}=\mathbf{r}_<$, $V'=V^<$:

$$U(\mathbf{r}_<)=\int_{V^<}G(\mathbf{r}_<,\mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')d^3r'-\frac{1}{4\pi}\Sigma^-(\mathbf{r}_<), \tag{4.10}$$

(2) $\mathbf{r}=\mathbf{r}_<$, $V'=V^>$:

$$0=\frac{1}{4\pi}\Sigma^-(\mathbf{r}_<)-\frac{1}{4\pi}\Sigma^\infty(\mathbf{r}_<), \tag{4.11}$$

(3) $\mathbf{r}=\mathbf{r}_>$, $V'=V^>$:

$$U(\mathbf{r}_>)=\frac{1}{4\pi}\Sigma^+(\mathbf{r}_>)-\frac{1}{4\pi}\Sigma^\infty(\mathbf{r}_>), \tag{4.12}$$

(4) $\mathbf{r}=\mathbf{r}_>$, $V'=V^<$:

$$0=\int_{V^<}G(\mathbf{r}_>,\mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')d^3r'-\frac{1}{4\pi}\Sigma^-(\mathbf{r}_>). \tag{4.13}$$

Here

$$\Sigma^\pm(\mathbf{r})\equiv\int_{S^\pm}\left(U(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}-G(\mathbf{r},\mathbf{r}')\frac{\partial U(\mathbf{r}')}{\partial n}\right)dS, \tag{4.14}$$

where S^+ and S^- are the inside limit and the outside limit from the surface S, respectively, and

$$\Sigma^\infty(\mathbf{r})\equiv\lim_{R\to\infty}\int_{S^{(R)}}\left(U(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n_R}-G(\mathbf{r},\mathbf{r}')\frac{\partial U(\mathbf{r}')}{\partial n_R}\right)dS. \tag{4.15}$$

In Eq. (4.15), $S^{(R)}$ denotes the outer boundary of the exterior region $\tilde{V}$ surrounding the scatterer, chosen for convenience to be a sphere of radius R, taken in the limit of $R\to\infty$. Further, $\partial/\partial n_R$ denotes differentiation along the outward radial direction.

The surface integral Σ^∞ may be readily evaluated and its value does not depend on the form of the potential. To see this, let us express $U(\mathbf{r})$ as the sum of an incident field $U^{(i)}$ and a scattered field $U^{(s)}$:

$$U(\mathbf{r})=U^{(i)}(\mathbf{r})+U^{(s)}(\mathbf{r}). \tag{4.16}$$

The scattered field behaves at infinity as an outgoing spherical wave and consequently, as is well known,

$$\lim_{R\to\infty}\int_{S^{(R)}}\left(U^{(s)}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n_R}-G(\mathbf{r},\mathbf{r}')\frac{\partial U^{(s)}(\mathbf{r}')}{\partial n_R}\right)dS=0. \tag{4.17}$$

On the other hand the incident field $U^{(i)}(\mathbf{r})$ obeys the Helmholtz equation throughout the whole space,

$$(\nabla^2+k^2)U^{(i)}(\mathbf{r})=0, \tag{4.18}$$

and it follows from Eq. (4.8) with $F=0$ that

$$\frac{1}{4\pi}\Sigma^\infty(\mathbf{r})=-U^{(i)}(\mathbf{r}), \tag{4.19}$$

for both $\mathbf{r}=\mathbf{r}_<$ and $\mathbf{r}=\mathbf{r}_>$. On substituting Eq. (4.19) into Eqs. (4.11) and (4.12) we find that

$$\frac{1}{4\pi}\Sigma^{S_+}(\mathbf{r}_<)=-U^{(i)}(\mathbf{r}_<) \tag{4.20}$$

and

$$U(\mathbf{r}_>)=\frac{1}{4\pi}\Sigma^{S_+}(\mathbf{r}_>)+U^{(i)}(\mathbf{r}_>). \tag{4.21}$$

Note that the formula (4.20) has the form of the extinction theorem (see also Ref. [7]).

Combining Eqs. (4.10), (4.11), and (4.19) we obtain an expression for the field inside the scattering volume:

$$U(\mathbf{r}_<)=\int_{V^<}G(\mathbf{r}_<,\mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')d^3r' +\frac{1}{4\pi}[\Sigma^{S_+}(\mathbf{r}_<)-\Sigma^{S_-}(\mathbf{r}_<)]+U^{(i)}(\mathbf{r}_<). \tag{4.22}$$

Similarly, from Eqs. (4.13) and (4.21) we obtain an expression for the field outside the scattering volume V:

$$U(\mathbf{r}_>)=\int_{V^<}G(\mathbf{r}_>,\mathbf{r}')F(\mathbf{r}')U(\mathbf{r}')d^3r' +\frac{1}{4\pi}[\Sigma^+(\mathbf{r}_>)-\Sigma^-(\mathbf{r}_>)]+U^{(i)}(\mathbf{r}_>). \tag{4.23}$$

From Eqs. (4.22) and (4.23) we see that if the field and its derivative are both assumed to be continuous across the surface S, then these expressions reduce to the usual results for the scattered field.

Let us consider the terms between brackets in Eqs. (4.22) and (4.23). They can be written as

$$\Sigma^+(\mathbf{r}_<)-\Sigma^-(\mathbf{r}_<)=\int_S\left\{\Delta U\frac{\partial G(\mathbf{r}_<,\mathbf{r}')}{\partial n} -G(\mathbf{r}_<,\mathbf{r}')\Delta\left(\frac{\partial U}{\partial n}\right)\right\}dS, \tag{4.24}$$

and

$$\Sigma^+(\mathbf{r}_>)-\Sigma^-(\mathbf{r}_>)=\int_S\left\{\Delta U\frac{\partial G(\mathbf{r}_>,\mathbf{r}')}{\partial n} -G(\mathbf{r}_>,\mathbf{r}')\Delta\left(\frac{\partial U}{\partial n}\right)\right\}dS, \tag{4.25}$$

respectively, where we have used the definitions (3.1) and (3.2). Equations (4.24) and (4.25) express the effect of the field discontinuities ΔU and $\Delta(\partial U/\partial n)$ on the scattered field. We briefly note the form that Eq. (4.23) takes when the observation point $\mathbf{r}_>$ is in the far zone.

Let $\hat{\mathbf{r}}$ be the unit vector in the direction of the observation point $\mathbf{r}_>=r\hat{\mathbf{r}}$. In the far zone, as $kr\to\infty$ with $\hat{\mathbf{r}}$ fixed, we have

$$|\mathbf{r}_>-\mathbf{r}'|\approx r-\mathbf{r}'\cdot\hat{\mathbf{r}}. \tag{4.26}$$

On substituting this approximation in the expression (4.7) for the outgoing Green function we find that

$$G(\mathbf{r}_>,\mathbf{r}')\approx\frac{e^{ikr}}{r}e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}}\quad(kr\to\infty), \tag{4.27}$$

$$\frac{\partial G(\mathbf{r}_>,\mathbf{r}')}{\partial n} \approx \hat{\mathbf{n}}\cdot\hat{\mathbf{r}} ik \frac{e^{ikr}}{r} e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}} \quad (kr\to\infty). \tag{4.28}$$

To analyze the far field we use the approximations (4.26)–(4.28) in Eq. (4.23) and find that

$$U(\mathbf{r}_>) \sim U^{(i)}(\mathbf{r}_>) + \frac{e^{ikr}}{r}\int_{V^<} e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}} F(\mathbf{r}')U(\mathbf{r}')d^3r' + \frac{1}{4\pi}\frac{e^{ikr}}{r}\int_S e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}}\left[ik\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\Delta U - \Delta\left(\frac{\partial U}{\partial n}\right)\right]dS. \tag{4.29}$$

As is to be expected, not only the volume integral but also the surface integral contributes to the far field.

V. CONCLUSIONS

We have shown how the usual approximate scalar theory for scattering of electromagnetic waves can be extended to take discontinuities of the field at the boundary of the scatterer into account. This extension takes the form of an additional integral over the surface of the scatterer. The integral contains jumps of both the field and its normal derivative. It is shown that in some cases the contribution of this surface integral to the far field can be significant.

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